

The Perron method for p -harmonic functions in unbounded sets in \mathbb{R}^n and metric spaces

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Abstract. The Perron method for solving the Dirichlet problem for p -harmonic functions is extended to unbounded open sets in the setting of a complete metric space with a doubling measure supporting a p -Poincaré inequality, $1 < p < \infty$. The upper and lower (p -harmonic) Perron solutions are studied for open sets, which are assumed to be p -parabolic if unbounded. It is shown that continuous functions and quasicontinuous Dirichlet functions are resolute (i.e., that their upper and lower Perron solutions coincide), that the Perron solution agrees with the p -harmonic extension, and that Perron solutions are invariant under perturbation of the function on a set of capacity zero.

Key words and phrases: Dirichlet problem, Dirichlet space, doubling measure, metric space, minimal p -weak upper gradient, Newtonian space, nonlinear potential theory, obstacle problem, p -harmonic, p -parabolic, Perron method, Poincaré inequality, quasicontinuity, upper gradient.

Mathematics Subject Classification (2010): Primary: 31E05; Secondary: 31C45, 35D30, 35J20, 35J25, 35J60, 47J20, 49J40, 49J52, 49Q20, 58J05, 58J32.

1. Introduction

The Dirichlet (boundary value) problem for p -harmonic functions, $1 < p < \infty$, which is a nonlinear generalization of the classical Dirichlet problem, considers the p -Laplace equation,

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad (1.1)$$

with prescribed boundary values $u = f$ on the boundary $\partial\Omega$. A continuous weak solution of (1.1) is said to be p -harmonic.

The nonlinear potential theory of p -harmonic functions has been developed since the 1960s; not only in \mathbb{R}^n , but also in weighted \mathbb{R}^n , Riemannian manifolds, and other settings. The books Malý–Ziemer [28] and Heinonen–Kilpeläinen–Martio [18] are two thorough treatments in \mathbb{R}^n and weighted \mathbb{R}^n , respectively.

More recently, p -harmonic functions have been studied in complete metric spaces equipped with a doubling measure supporting a p -Poincaré inequality. It is not clear how to employ partial differential equations in such a general setting as a metric measure space. However, the equivalent variational problem of locally minimizing the p -energy integral,

$$\int |\nabla u|^p dx, \quad (1.2)$$

among all admissible functions, becomes available when considering the notion of minimal p -weak upper gradient as a substitute for the modulus of the usual gradient. A continuous minimizer of (1.2) is p -harmonic. The reader might want to consult

Björn–Björn [3] for the theory of p -harmonic functions and first-order analysis on metric spaces.

If the boundary value function f is not continuous, then it is not feasible to require that the solution u attains the boundary values as limits, i.e., to require that $u(y) \rightarrow f(x)$ as $y \rightarrow x$ ($y \in \Omega$) for all $x \in \partial\Omega$. This is actually often not possible even if f is continuous (see, e.g., Examples 13.3 and 13.4 in Björn–Björn [3]). It is therefore more reasonable to consider boundary data in a weaker (Sobolev) sense. Shanmugalingam [33] solved the Dirichlet problem for p -harmonic functions in bounded domains with Newtonian boundary data taken in Sobolev sense. This result was generalized by Hansevi [16] to unbounded domains with Dirichlet boundary data. For continuous boundary values, the problem was solved in bounded domains using uniform approximation by Björn–Björn–Shanmugalingam [6].

The Perron method for solving the Dirichlet problem for harmonic functions (on \mathbb{R}^2) was introduced in 1923 by Perron [29] (and independently by Remak [30]). The advantage of the method is that one can construct reasonable solutions for arbitrary boundary data. It provides an upper and a lower solution, and the major question is to determine when these solutions coincide, i.e., to determine when the boundary data is *resolutive*. The Perron method in connection with the usual Laplace operator has been studied extensively in Euclidean domains (see, e.g., Brelot [11] for the complete characterization of the resolutive functions) and has been extended to degenerate elliptic operators (see, e.g., Granlund–Lindqvist–Martio [14], Kilpeläinen [23], and Heinonen–Kilpeläinen–Martio [18]).

Björn–Björn–Shanmugalingam [7] extended the Perron method for p -harmonic functions to the setting of a complete metric space equipped with a doubling measure supporting a p -Poincaré inequality, and proved that Perron solutions are p -harmonic and agree with the previously obtained solutions for Newtonian boundary data in Shanmugalingam [33]. More recently, Björn–Björn–Shanmugalingam [9] have developed the Perron method for p -harmonic functions with respect to the Mazurkiewicz boundary. See also Estep–Shanmugalingam [12], A. Björn [2], and Björn–Björn–Sjödín [10].

The purpose of this paper is to extend the Perron method for solving the Dirichlet problem for p -harmonic functions to *unbounded* open sets in the setting of a complete metric space equipped with a doubling measure supporting a p -Poincaré inequality. In particular, we show that quasicontinuous functions with finite Dirichlet energy, as well as continuous functions, are resolutive with respect to open sets, which are assumed to be p -parabolic if unbounded, and that the Perron solution is the unique p -harmonic solution that takes the required boundary data outside sets of capacity zero. We also show that Perron solutions are invariant under perturbations on sets of capacity zero.

The paper is organized as follows: In the next section, we establish notation, review some basic definitions relating to Sobolev-type spaces on metric spaces, and obtain a new convergence lemma. In Section 3, we review the obstacle problem associated with p -harmonic functions in unbounded sets and obtain a convergence theorem that will be important in the proof of Theorem 7.5 (the main result of this paper). Section 4 is devoted to p -parabolic sets. The necessary background on p -harmonic and superharmonic functions is given in Section 5, making it possible to define Perron solutions in Section 6, where we also extend the comparison principle for superharmonic functions to unbounded sets. In Section 7, we introduce a smaller capacity (and its related quasicontinuity property) before we obtain our main result (Theorem 7.5) on resolutive (of quasicontinuous functions) along with some consequences.

2. Notation and preliminaries

We assume throughout the paper that (X, \mathcal{M}, μ, d) is a metric measure space (which we refer to as X) equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We use the following notation for balls,

$$B(x_0, r) := \{x \in X : d(x, x_0) < r\},$$

and for $B = B(x_0, r)$ and $\lambda > 0$, we let $\lambda B = B(x_0, \lambda r)$. The σ -algebra \mathcal{M} (on which μ is defined) is the completion of the Borel σ -algebra. Later we will impose additional requirements on the space and on the measure. We assume further that $1 < p < \infty$ and that Ω is a nonempty (possibly unbounded) open subset of X .

The measure μ is said to be *doubling* if there exists a constant $C \geq 1$ such that

$$0 < \mu(2B) \leq C\mu(B) < \infty$$

for all balls $B \subset X$. Recall that a metric space is said to be *proper* if all bounded closed subsets are compact. In particular, this is true if the metric space is complete and the measure is doubling.

The characteristic function of a set E is denoted by χ_E , and we let $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. We say that the set E is compactly contained in A if \overline{E} (the closure of E) is a compact subset of A and denote this by $E \Subset A$. The extended real number system is denoted by $\overline{\mathbb{R}} := [-\infty, \infty]$. We use the notation $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Continuous functions will be assumed to be real-valued. By a curve in X we mean a rectifiable nonconstant continuous mapping from a compact interval into X . A curve can thus be parametrized by its arc length ds .

Definition 2.1. A Borel function $g: X \rightarrow [0, \infty]$ is said to be an *upper gradient* of a function $f: X \rightarrow \overline{\mathbb{R}}$ whenever

$$|f(x) - f(y)| \leq \int_{\gamma} g \, ds \tag{2.1}$$

holds for each pair of points $x, y \in X$ and every curve γ in X joining x and y . We make the convention that the left-hand side is infinite when at least one of the terms in the left-hand side is infinite.

A drawback of the upper gradients, introduced in Heinonen–Koskela [19],[20], is that they are not preserved by L^p -convergence. It is, however, possible to overcome this problem by relaxing the condition a bit (Koskela–MacManus [27]).

Definition 2.2. A measurable function $g: X \rightarrow [0, \infty]$ is said to be a *p -weak upper gradient* of a function $f: X \rightarrow \overline{\mathbb{R}}$ whenever (2.1) holds for each pair of points $x, y \in X$ and p -almost every curve (see below) γ in X joining x and y .

Note that a p -weak upper gradient is not required to be a Borel function (see the discussion in the notes to Chapter 1 in Björn–Björn [3]).

We say that a property holds for *p -almost every curve* if it fails only for a curve family Γ with zero p -modulus, i.e., if there exists a nonnegative $\rho \in L^p(X)$ such that $\int_{\gamma} \rho \, ds = \infty$ for every curve $\gamma \in \Gamma$.

A countable union of curve families, each with zero p -modulus, also has zero p -modulus. For proofs of this and other results in this section, we refer to Björn–Björn [3] or Heinonen–Koskela–Shanmugalingam–Tyson [21].

Shanmugalingam [32] used upper gradients to define so-called Newtonian spaces.

Definition 2.3. The *Newtonian space* on X , denoted by $N^{1,p}(X)$, is the space of all everywhere defined, extended real-valued functions $u \in L^p(X)$ such that

$$\|u\|_{N^{1,p}(X)} := \left(\int_X |u|^p d\mu + \inf_g \int_X g^p d\mu \right)^{1/p} < \infty,$$

where the infimum is taken over all upper gradients g of u .

Definition 2.4. An everywhere defined, measurable, extended real-valued function on X belongs to the *Dirichlet space* $D^p(X)$ if it has an upper gradient in $L^p(X)$.

It follows from Lemma 2.4 in Koskela–MacManus [27] that a measurable function belongs to $D^p(X)$ whenever it (merely) has a p -weak upper gradient in $L^p(X)$.

We emphasize that Newtonian and Dirichlet functions are defined *everywhere* (not just up to an equivalence class in the corresponding function space), which is essential for the notion of upper gradient to make sense. Shanmugalingam [32] proved that the associated normed (quotient) space defined by $N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, is a Banach space.

A measurable set $A \subset X$ can be considered to be a metric space in its own right (with the restriction of d and μ to A). Thus the Newtonian space $N^{1,p}(A)$ and the Dirichlet space $D^p(A)$ are also given by Definitions 2.3 and 2.4, respectively. If X is proper, then $f \in L^p_{\text{loc}}(\Omega)$, $f \in N^{1,p}_{\text{loc}}(\Omega)$, and $f \in D^p_{\text{loc}}(\Omega)$ if and only if $f \in L^p(\Omega')$, $f \in N^{1,p}(\Omega')$, and $f \in D^p(\Omega')$, respectively, for all open $\Omega' \Subset \Omega$.

If $u \in D^p(X)$, then u has a *minimal p -weak upper gradient*, denoted by g_u , which is minimal in the sense that $g_u \leq g$ a.e. for all p -weak upper gradients g of u ; see Shanmugalingam [33]. Minimal p -weak upper gradients g_u are true substitutes for $|\nabla u|$ in metric spaces. One of the important properties of minimal p -weak upper gradients is that they are local in the sense that if two functions $u, v \in D^p(X)$ coincide on a set E , then $g_u = g_v$ a.e. on E . Furthermore, if $U = \{x \in X : u(x) > v(x)\}$, then $g_u \chi_U + g_v \chi_{X \setminus U}$ and $g_v \chi_U + g_u \chi_{X \setminus U}$ are minimal p -weak upper gradients of $\max\{u, v\}$ and $\min\{u, v\}$, respectively. The restriction of a minimal p -weak upper gradient to an open subset remains minimal with respect to that subset, and hence the results above about minimal p -weak upper gradients of functions in $D^p(X)$ extend to functions in $D^p_{\text{loc}}(X)$ having minimal p -weak upper gradients in $L^p_{\text{loc}}(X)$.

The notion of capacity of a set is important in potential theory, and various types and definitions can be found in the literature (see, e.g., Kinnunen–Martio [24] and Shanmugalingam [32]).

Definition 2.5. Let $A \subset X$ be measurable. The (*Sobolev*) *capacity* (with respect to A) of $E \subset A$ is the number

$$C_p(E; A) := \inf_u \|u\|_{N^{1,p}(A)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(A)$ such that $u \geq 1$ on E . When the capacity is taken with respect to X , we simplify the notation and write $C_p(E)$.

Whenever a property holds for all points except for those in a set of capacity zero, it is said to hold *quasieverywhere* (q.e.).

The capacity is countably subadditive, i.e., $C_p(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} C_p(E_j)$.

In order to be able to compare boundary values of Dirichlet and Newtonian functions, we introduce the following spaces.

Definition 2.6. For subsets E and A of X , where A is measurable, the *Dirichlet space with zero boundary values in $A \setminus E$* , is

$$D_0^p(E; A) := \{u|_{E \cap A} : u \in D^p(A) \text{ and } u = 0 \text{ in } A \setminus E\}.$$

The *Newtonian space with zero boundary values*, $N_0^{1,p}(E; A)$, is defined analogously. We let $D_0^p(E)$ and $N_0^{1,p}(E)$ denote $D_0^p(E; X)$ and $N_0^{1,p}(E; X)$, respectively.

The condition “ $u = 0$ in $A \setminus E$ ” can actually be replaced by “ $u = 0$ q.e. in $A \setminus E$ ” without changing the obtained spaces.

If $E \subset X$ is measurable, $f \in D^p(E)$, $f_1, f_2 \in D_0^p(E)$, and $f_1 \leq f \leq f_2$ q.e. in E , then $f \in D_0^p(E)$ (this is Lemma 2.8 in Hansevi [16]).

The following convergence lemma will be used to prove Theorem 3.2, which in turn will be important when we prove Theorem 7.5.

Lemma 2.7. *Let G_1, G_2, \dots be open sets such that $G_1 \subset G_2 \subset \dots \subset X = \bigcup_{k=1}^{\infty} G_k$ and let $\{u_j\}_{j=1}^{\infty}$ be a sequence of functions defined on X . Assume that $\{u_j\}_{j=1}^{\infty}$ is bounded in $L^p(G_k)$ for all $k = 1, 2, \dots$. Assume further that $\{g_j\}_{j=1}^{\infty}$ is bounded in $L^p(X)$, and that g_j is a p -weak upper gradient of u_j with respect to G_j for each $j = 1, 2, \dots$. Then a function u belongs to $D^p(X)$ if $u_j \rightarrow u$ q.e. on X as $j \rightarrow \infty$.*

Proof. Let k be a positive integer. Clearly, g_j is a p -weak upper gradient of u_j with respect to G_k for every integer $j \geq k$. According to Lemma 3.2 in Björn–Björn–Parviainen [5], there are a p -weak upper gradient $\tilde{g}_k \in L^p(G_k)$ of u with respect to G_k and a subsequence of $\{g_j\}_{j=1}^{\infty}$, denoted by $\{g_{k,j}\}_{j=1}^{\infty}$, such that $g_{k,j} \rightarrow \tilde{g}_k$ weakly in $L^p(G_k)$ as $j \rightarrow \infty$. Extend \tilde{g}_k to X by letting $\tilde{g}_k = 0$ on $X \setminus G_k$. Since $\{g_j\}_{j=1}^{\infty}$ is bounded in $L^p(X)$, there is an integer M such that $\|g_j\|_{L^p(X)} \leq M$ for all $j = 1, 2, \dots$. The weak convergence implies that

$$\|\tilde{g}_k\|_{L^p(X)} = \|\tilde{g}_k\|_{L^p(G_k)} \leq \liminf_{j \rightarrow \infty} \|g_{k,j}\|_{L^p(G_k)} \leq \liminf_{j \rightarrow \infty} \|g_{k,j}\|_{L^p(X)} \leq M,$$

and hence the sequence $\{\tilde{g}_k\}_{k=1}^{\infty}$ is bounded in $L^p(X)$.

Since $L^p(X)$ is reflexive, it follows from Banach–Alaoglu’s theorem that there is a subsequence, also denoted by $\{\tilde{g}_k\}_{k=1}^{\infty}$, that converges weakly in $L^p(X)$ to a function g . By applying Mazur’s lemma (see, e.g., Theorem 3.12 in Rudin [31]) repeatedly to the sequences $\{\tilde{g}_k\}_{k=j}^{\infty}$, $j = 1, 2, \dots$, we can find convex combinations

$$g'_j = \sum_{k=j}^{N_j} a_{j,k} \tilde{g}_k$$

such that $\|g'_j - g\|_{L^p(X)} < 1/j$, and hence we obtain a sequence $\{g'_j\}_{j=1}^{\infty}$ that converges to g in $L^p(X)$. Note that $g \in L^p(X)$, and that for every $n = 1, 2, \dots$, the sequence $\{g'_j\}_{j=n}^{\infty}$ consists of p -weak upper gradients of u with respect to G_n . It suffices to show that g is a p -weak upper gradient of u to complete the proof.

By Fuglede’s lemma (Lemma 3.4 in Shanmugalingam [32]), we can find a subsequence, also denoted by $\{g'_j\}_{j=1}^{\infty}$, and a collection of curves Γ in X with zero p -modulus, such that for every curve $\gamma \notin \Gamma$, it follows that

$$\int_{\gamma} g'_j ds \rightarrow \int_{\gamma} g ds \quad \text{as } j \rightarrow \infty. \quad (2.2)$$

For every $n = 1, 2, \dots$, let $\Gamma_{n,j}$, $j = n, n+1, \dots$, be the collection of curves in G_n along which g'_j is not an upper gradient of u , and let

$$\Gamma' = \Gamma \cup \bigcup_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \Gamma_{n,j}.$$

Then Γ' has zero p -modulus.

Let $\gamma \notin \Gamma'$ be an arbitrary curve in X with endpoints x and y . Since γ is compact and G_1, G_2, \dots are open sets that exhaust X , we can find an integer N such that $\gamma \subset G_N$ and

$$|u(x) - u(y)| \leq \int_{\gamma} g'_j ds, \quad j = N, N+1, \dots$$

It follows that g is a p -weak upper gradient of u , and thus $u \in D^p(X)$, since

$$|u(x) - u(y)| \leq \lim_{j \rightarrow \infty} \int_{\gamma} g'_j ds = \int_{\gamma} g ds. \quad \square$$

Definition 2.8. Let $q \geq 1$. We say that X supports a (q, p) -Poincaré inequality if there exist constants, $C > 0$ and $\lambda \geq 1$ (the dilation constant), such that

$$\left(\int_B |u - u_B|^q d\mu \right)^{1/q} \leq C \operatorname{diam}(B) \left(\int_{\lambda B} g^p d\mu \right)^{1/p} \quad (2.3)$$

for all balls $B \subset X$, all integrable functions u on X , and all upper gradients g of u .

In (2.3), we have used the convenient notation $u_B := \int_B u d\mu / \mu(B) := \frac{1}{\mu(B)} \int_B u d\mu$. We usually write p -Poincaré inequality instead of $(1, p)$ -Poincaré inequality.

Requiring a Poincaré inequality to hold is one way of making it possible to control functions by their upper gradients.

3. The obstacle problem

In this section, we also assume that X is proper and supports a (p, p) -Poincaré inequality, and that $C_p(X \setminus \Omega) > 0$.

Inspired by Kinnunen–Martio [25], the following obstacle problem, which is a generalization that allows for unbounded sets, was defined in Hansevi [16].

Definition 3.1. Let $V \subset X$ be a nonempty open subset with $C_p(X \setminus V) > 0$. For $\psi: V \rightarrow \overline{\mathbb{R}}$ and $f \in D^p(V)$, define

$$\mathcal{K}_{\psi, f}(V) = \{v \in D^p(V) : v - f \in D_0^p(V) \text{ and } v \geq \psi \text{ q.e. in } V\}.$$

A function u is said to be a *solution of the $\mathcal{K}_{\psi, f}(V)$ -obstacle problem (with obstacle ψ and boundary values f)* whenever $u \in \mathcal{K}_{\psi, f}(V)$ and

$$\int_V g_u^p d\mu \leq \int_V g_v^p d\mu \quad \text{for all } v \in \mathcal{K}_{\psi, f}(V).$$

When $V = \Omega$, we usually denote $\mathcal{K}_{\psi, f}(\Omega)$ by $\mathcal{K}_{\psi, f}$ for short.

It was proved in Hansevi [16] that the $\mathcal{K}_{\psi, f}$ -obstacle problem has a unique (up to sets of capacity zero) solution under the natural condition of $\mathcal{K}_{\psi, f}$ being nonempty. If the measure μ is doubling, then there is a unique lsc-regularized solution of the $\mathcal{K}_{\psi, f}$ -obstacle problem whenever $\mathcal{K}_{\psi, f}$ is nonempty (Theorem 4.1 in Hansevi [16]). The *lsc-regularization* of u is the (lower semicontinuous) function u^* defined by

$$u^*(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y) := \lim_{r \rightarrow 0} \operatorname{ess\,inf}_{B(x, r)} u.$$

We conclude this section with a proof of a new convergence theorem that will be used in the proof of Theorem 7.5. It is a generalization of Proposition 10.18 in Björn–Björn [3] to unbounded sets and Dirichlet functions. The special case when $\psi_j = f_j \in N^{1,p}(\Omega)$ had previously been proved in Kinnunen–Shanmugalingam [26], and a similar result for the double obstacle problem was obtained in Farnana [13].

Theorem 3.2. *Let $\{\psi_j\}_{j=1}^\infty$ and $\{f_j\}_{j=1}^\infty$ be sequences of functions in $D^p(\Omega)$ that are decreasing q.e. to functions ψ and f in $D^p(\Omega)$, respectively, and are such that $\|g_{\psi_j-\psi}\|_{L^p(\Omega)} \rightarrow 0$ and $\|g_{f_j-f}\|_{L^p(\Omega)} \rightarrow 0$ as $j \rightarrow \infty$. If u_j is a solution of the $\mathcal{K}_{\psi_j, f_j}$ -obstacle problem for each $j = 1, 2, \dots$, then the sequence $\{u_j\}_{j=1}^\infty$ is decreasing q.e. in Ω to a function which is a solution of the $\mathcal{K}_{\psi, f}$ -obstacle problem.*

Proof. The comparison principle (Lemma 3.6 in Hansevi [16]) asserts that $u_{j+1} \leq u_j$ q.e. in Ω for each $j = 1, 2, \dots$, and hence by the subadditivity of the capacity there exists a function u such that $\{u_j\}_{j=1}^\infty$ is decreasing to u q.e. in Ω . We will show that u is a solution of the $\mathcal{K}_{\psi, f}$ -obstacle problem.

Let $w_j = u_j - f_j$ and $w = u - f$, all functions extended by zero outside Ω . Let $B \subset X$ be a ball such that $B \cap \Omega$ is nonempty and $C_p(B' \setminus \Omega) > 0$ where $B' := \frac{1}{2}B$.

We claim that the sequences $\{g_{w_j}\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ are bounded in $L^p(X)$ and $L^p(kB)$, respectively, for every $k = 1, 2, \dots$. To show this, let k be a positive integer. Let $S = \bigcap_{j=1}^\infty S_j$, where $S_j := \{x \in X : w_j(x) = 0\}$. Proposition 4.14 in Björn–Björn [3] asserts that $w_j \in N_{\text{loc}}^{1,p}(X)$, and since

$$C_p(kB' \cap S_j) \geq C_p(kB' \cap S) \geq C_p(kB' \setminus \Omega) \geq C_p(B' \setminus \Omega) > 0,$$

Maz'ya's inequality (Theorem 5.53 in Björn–Björn [3]) implies the existence of constants $C_{kB, \Omega} > 0$ and $\lambda \geq 1$ such that

$$\int_{kB} |w_j|^p d\mu \leq C_{kB, \Omega} \int_{\lambda kB} g_{w_j}^p d\mu.$$

Let $h_j = \max\{f_j, \psi_j\}$. Then $0 \leq h_j - f_j = (\psi_j - f_j)_+ \leq (u_j - f_j)_+$ q.e. in Ω , and hence Lemma 2.8 in Hansevi [16] asserts that $h_j - f_j \in D_0^p(\Omega)$. Clearly, $h_j \in \mathcal{K}_{\psi_j, f_j}$, and since u_j is a solution of the $\mathcal{K}_{\psi_j, f_j}$ -obstacle problem, it follows that $\|g_{u_j}\|_{L^p(\Omega)} \leq \|g_{h_j}\|_{L^p(\Omega)}$. We also know that $g_{h_j} \leq g_{\psi_j} + g_{f_j}$ a.e. in Ω , and therefore the claim follows because

$$\begin{aligned} C_{kB, \Omega}^{-1/p} \|w_j\|_{L^p(kB)} &\leq \|g_{w_j}\|_{L^p(X)} \\ &\leq \|g_{u_j}\|_{L^p(\Omega)} + \|g_{f_j}\|_{L^p(\Omega)} \\ &\leq \|g_{h_j}\|_{L^p(\Omega)} + \|g_{f_j}\|_{L^p(\Omega)} \\ &\leq \|g_{\psi_j}\|_{L^p(\Omega)} + 2\|g_{f_j}\|_{L^p(\Omega)} \\ &\leq \|g_{\psi_j-\psi}\|_{L^p(\Omega)} + \|g_{\psi}\|_{L^p(\Omega)} + 2\|g_{f_j-f}\|_{L^p(\Omega)} + 2\|g_f\|_{L^p(\Omega)}. \end{aligned} \quad (3.1)$$

Lemma 2.7 applies here and asserts that $w \in D^p(X)$, and hence $u - f \in D_0^p(\Omega)$. Because $f \in D^p(\Omega)$, this also shows that $u \in D^p(\Omega)$. Since C_p is countably sub-additive, $u \geq \psi$ q.e. in Ω , and hence $u \in \mathcal{K}_{\psi, f}$.

Let v be an arbitrary function that belongs to $\mathcal{K}_{\psi, f}$. We complete the proof by showing that

$$\int_{\Omega} g_u^p d\mu \leq \int_{\Omega} g_v^p d\mu. \quad (3.2)$$

Let $\varphi_j = \max\{v + f_j - f, \psi_j\}$. Clearly, $\varphi_j \geq \psi_j$ and $\varphi_j \in D^p(\Omega)$. Furthermore,

$$v - f \leq \max\{v - f, \psi_j - f_j\} = \varphi_j - f_j \leq \max\{v - f, (u_j - f_j)_+\} \quad \text{q.e. in } \Omega,$$

and hence $\varphi_j - f_j \in D_0^p(\Omega)$ by Lemma 2.8 in Hansevi [16]. We conclude that $\varphi_j \in \mathcal{K}_{\psi_j, f_j}$, and therefore

$$\int_{\Omega} g_{u_j}^p d\mu \leq \int_{\Omega} g_{\varphi_j}^p d\mu.$$

Let E be the set where $\{f_j\}_{j=1}^\infty$ decreases to f , $\{\psi_j\}_{j=1}^\infty$ decreases to ψ , and simultaneously $v \geq \psi$. Then $C_p(\Omega \setminus E) = 0$.

Let $U_j = \{x \in E : (f_j - f)(x) < (\psi_j - v)(x)\}$. Clearly, $\varphi_j - v = \psi_j - v$ in U_j and $\varphi_j - v = f_j - f$ in $E \setminus U_j$, and hence it follows that

$$\begin{aligned} \int_{\Omega} g_{\varphi_j - v}^p d\mu &\leq \int_{U_j} (g_{\psi_j - \psi} + g_{\psi - v})^p d\mu + \int_{E \setminus U_j} g_{f_j - f}^p d\mu \\ &\leq 2^p \int_{U_j} g_{\psi - v}^p d\mu + 2^p \int_{\Omega} g_{\psi_j - \psi}^p d\mu + \int_{\Omega} g_{f_j - f}^p d\mu, \end{aligned} \quad (3.3)$$

where the last two integrals tend to zero as $j \rightarrow \infty$.

Let $V_j = \{x \in E : \psi(x) < v(x) < \psi_j(x)\}$. Since $f_j - f \geq 0$ in E , we know that $v < \psi_j$ in U_j , and because $g_{\psi - v} = 0$ a.e. in

$$\{x \in E : v(x) \leq \psi(x)\} = \{x \in E : v(x) = \psi(x)\},$$

it follows that

$$\int_{U_j} g_{\psi - v}^p d\mu \leq \int_{V_j} g_{\psi - v}^p d\mu. \quad (3.4)$$

The fact that $\{\psi_j\}_{j=1}^{\infty}$ is decreasing to ψ in E implies that $g_{\psi - v} \chi_{V_j} \rightarrow 0$ everywhere in E as $j \rightarrow \infty$, and since $|g_{\psi - v} \chi_{V_j}| \leq g_{\psi - v} \leq g_{\psi} + g_v$ a.e. in E and $g_{\psi} + g_v \in L^p(E)$, dominated convergence asserts that

$$\int_{V_j} g_{\psi - v}^p d\mu = \int_E g_{\psi - v}^p \chi_{V_j} d\mu \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (3.5)$$

It follows from (3.3), (3.4), and (3.5) that $g_{\varphi_j} \rightarrow g_v$ in $L^p(\Omega)$ as $j \rightarrow \infty$.

Let

$$\Omega_k = \{x \in kB \cap \Omega : \text{dist}(x, \partial\Omega) > \delta/k\}, \quad k = 1, 2, \dots,$$

where $\delta > 0$ is sufficiently small so that Ω_1 is nonempty. It is clear that

$$\Omega_1 \Subset \Omega_2 \Subset \dots \Subset \Omega = \bigcup_{k=1}^{\infty} \Omega_k.$$

Fix a positive integer k . Then g_u and g_{u_j} are minimal p -weak upper gradients of u and u_j , respectively, with respect to Ω_k . By Proposition 4.14 in Björn–Björn [3], the functions f and f_j belong to $L^p_{\text{loc}}(\Omega)$, and hence f and f_j are in $L^p(\Omega_k)$. Furthermore, $\{f_j\}_{j=1}^{\infty}$ is decreasing to f q.e. in Ω , and therefore $|f_j - f| \leq |f_1 - f|$ q.e. in Ω . By (3.1), we can see that $\{w_j\}_{j=1}^{\infty}$ is bounded in $L^p(kB)$, and also that $\{g_{u_j}\}_{j=1}^{\infty}$ is bounded in $L^p(\Omega)$. Since

$$\|u_j\|_{L^p(\Omega_k)} \leq \|w_j\|_{L^p(kB)} + \|f_1 - f\|_{L^p(\Omega_k)} + \|f\|_{L^p(\Omega_k)},$$

it follows that $\{u_j\}_{j=1}^{\infty}$ is bounded in $N^{1,p}(\Omega_k)$, and because $u_j \rightarrow u$ q.e. in Ω as $j \rightarrow \infty$, Corollary 3.3 in Björn–Björn–Parviainen [5] asserts that

$$\int_{\Omega_k} g_u^p d\mu \leq \liminf_{j \rightarrow \infty} \int_{\Omega_k} g_{u_j}^p d\mu \leq \liminf_{j \rightarrow \infty} \int_{\Omega} g_{u_j}^p d\mu \leq \liminf_{j \rightarrow \infty} \int_{\Omega} g_{\varphi_j}^p d\mu = \int_{\Omega} g_v^p d\mu.$$

Letting $k \rightarrow \infty$ yields (3.2) and the proof is complete. \square

If μ is doubling, then X is proper if and only if X is complete (see, e.g., Proposition 3.1 in Björn–Björn [3]). Hölder's inequality implies that X supports a p -Poincaré inequality if X supports a (p, p) -Poincaré inequality. The converse is true when μ is doubling; see Theorem 5.1 in Hajlasz–Koskela [15]. Thus adding the assumption that μ is doubling leads to the rather standard assumptions stated below.

We assume from now on that $1 < p < \infty$, that X is a complete metric measure space supporting a p -Poincaré inequality, that μ is doubling, and that $\Omega \subset X$ is a nonempty (possibly unbounded) open subset with $C_p(X \setminus \Omega) > 0$.

4. p -parabolicity

Note the standing assumptions described at the end of the previous section.

In the proof of Theorem 7.5, we need Ω to be p -parabolic if it is unbounded.

Definition 4.1. If Ω is unbounded, then we say that Ω is p -parabolic if for every compact $K \subset \Omega$, there exist functions $u_j \in N^{1,p}(\Omega)$ such that $u_j \geq 1$ on K for all $j = 1, 2, \dots$, and

$$\int_{\Omega} g_{u_j}^p d\mu \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (4.1)$$

Otherwise, Ω is said to be p -hyperbolic.

In Definition 4.1, we may as well use $u_j \in D^p(\Omega)$ with bounded support such that $\chi_K \leq u_j \leq 1$, $j = 1, 2, \dots$ (see, e.g., the proof of Lemma 5.43 in Björn–Björn [3]).

Remark 4.2. If $\Omega_1 \subset \Omega_2$, then Ω_1 is p -parabolic whenever Ω_2 is p -parabolic.

Holopainen–Shanmugalingam [22] proposed a definition of p -harmonic Green functions (i.e., fundamental solutions of the p -Laplace operator) on metric spaces. The functions they defined did, however, not share all characteristics with Green functions, and therefore they gave them another name; they called them p -singular functions. Theorem 3.14 in [22] asserts that if X is locally linearly locally connected (see Section 2 in [22] for the definition), then the space X is p -hyperbolic if and only if for every $y \in X$ there exists a p -singular function with singularity at y .

Example 4.3. The space \mathbb{R}^n , $n \geq 1$, is p -parabolic if and only if $p \geq n$. (It follows that all open subsets of \mathbb{R}^n are p -parabolic for all $p \geq n$; see Remark 4.2.)

To see this, assume that $p \geq n$ and let $K \subset \mathbb{R}^n$ be compact. Choose R sufficiently large so that $K \subset B := B(0, R)$. Let

$$u_j(x) = \min \left\{ 1, \left(1 - \frac{\log |x/R|}{j} \right)_+ \right\}, \quad j = 1, 2, \dots \quad (4.2)$$

Then $\{u_j\}_{j=1}^{\infty}$ is a sequence of admissible functions for (4.1), and

$$g_{u_j} = (j|x|)^{-1} \chi_{B_j \setminus B}, \quad j = 1, 2, \dots,$$

where $B_j := B(0, Re^j)$. It follows that

$$\int_{\mathbb{R}^n} g_{u_j}^p dx = C_n \int_R^{Re^j} \frac{r^{n-1}}{(jr)^p} dr = C_n \begin{cases} \frac{R^{n-p}(1 - e^{-j(p-n)})}{(p-n)j^p} & \text{if } p > n, \\ j^{1-p} & \text{if } p = n, \end{cases}$$

and hence $\int_{\mathbb{R}^n} g_{u_j}^p dx \rightarrow 0$ as $j \rightarrow \infty$.

The necessity follows from Theorem 3.14 in Holopainen–Shanmugalingam [22], because if we assume that $p < n$ and let $y \in \mathbb{R}^n$, then

$$f(x) = |x - y|^{\frac{p-n}{p-1}}, \quad x \in \mathbb{R}^n,$$

is a Green function with singularity at y that is p -harmonic in $\mathbb{R}^n \setminus \{y\}$.

A set can be p -parabolic if it does not “grow too much” towards infinity, even though the surrounding space is not p -parabolic.

Example 4.4. Let $n \geq 2$ and assume that $1 < p < n$. Let

$$\Omega_f = \{x = (x', \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x' < f(|\tilde{x}|)\},$$

where

$$f(r) \leq \begin{cases} C & \text{if } r < 1, \\ Cr^q & \text{if } r \geq 1, \end{cases}$$

and $q \leq p - n + 1$ (note that $q < 1$ since $p < n$).

Let $K \subset \Omega_f$ be compact. Choose R sufficiently large so that $K \subset B := B(0, R)$. It can be chosen large enough so that $|\tilde{x}| \geq R/2 \geq 1$ for all $(x', \tilde{x}) \in \Omega_f \setminus B$. This is possible since $q < 1$ and $f(r) < Cr^q$. Define the sequence of admissible functions $\{u_j\}_{j=1}^\infty$ as in (4.2). Then

$$\begin{aligned} \int_{\Omega_f} g_{u_j}^p dx &= \int_{\mathbb{R}^{n-1}} \int_0^{f(|\tilde{x}|)} \frac{\chi_{B_j \setminus B}}{(j|x|)^p} dx' d\tilde{x} \\ &\leq \frac{C_{n-1}}{j^p} \int_{R/2}^{Re^j} \frac{f(r)}{r^p} r^{n-2} dr = \frac{C'_{n-1}}{j^p} \int_{R/2}^{Re^j} r^{q-p+n-2} dr =: I_j. \end{aligned}$$

Since

$$\int_{R/2}^{Re^j} r^{q-p+n-2} dr = \begin{cases} j + \log 2 & \text{if } q = p - n + 1, \\ \frac{(e^{j(q-p+n-1)} - 2^{-(q-p+n-1)})R^{q-p+n-1}}{q - p + n - 1} & \text{if } q < p - n + 1, \end{cases}$$

it follows that $\int_{\Omega_f} g_{u_j}^p dx \leq I_j \rightarrow 0$ as $j \rightarrow \infty$. Thus Ω_f is p -parabolic (while \mathbb{R}^n is not p -parabolic since $p < n$ in this case).

5. p -harmonic and superharmonic functions

The standing assumptions are described at the end of Section 3.

There are many equivalent definitions of (super)minimizers (or, more accurately, p -(super)minimizers) in the literature (see, e.g., Proposition 3.2 in A. Björn [1]).

Definition 5.1. We say that a function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is a *superminimizer* in Ω if

$$\int_{\varphi \neq 0} g_u^p d\mu \leq \int_{\varphi \neq 0} g_{u+\varphi}^p d\mu \quad (5.1)$$

holds for all nonnegative $\varphi \in N_0^{1,p}(\Omega)$, and a *minimizer* in Ω if (5.1) holds for all $\varphi \in N_0^{1,p}(\Omega)$. Moreover, a function is p -harmonic if it is a continuous minimizer.

According to Proposition 3.2 in A. Björn [1], it is in fact only necessary to test (5.1) with (all nonnegative and all, respectively) $\varphi \in \text{Lip}_c(\Omega)$.

Proposition 3.9 in Hansevi [16] asserts that a function u is a superminimizer in Ω if u is a solution of the $\mathcal{H}_{\psi,f}$ -obstacle problem.

The following definition makes sense due to Theorem 4.4 in Hansevi [16]. Because Proposition 2.7 in Björn–Björn [4] asserts that $D_0^p(\Omega) = N_0^{1,p}(\Omega)$ if Ω is bounded, it is a generalization of Definition 8.31 in Björn–Björn [3] to Dirichlet functions and to unbounded sets.

Definition 5.2. Let $V \subset X$ be a nonempty open set with $C_p(X \setminus V) > 0$. The p -harmonic extension $H_V f$ of $f \in D^p(V)$ to V is the continuous solution of the $\mathcal{H}_{-\infty,f}(V)$ -obstacle problem. When $V = \Omega$ we usually write Hf instead of $H_\Omega f$.

If f is defined outside V , then we sometimes consider $H_V f$ to be equal to f in some set outside V where f is defined.

A Lipschitz function f on ∂V can be extended to a Lipschitz function \bar{f} on \bar{V} (see, e.g., Theorem 6.2 in Heinonen [17]), and $\bar{f} \in N^{1,p}(\bar{V})$ if V is bounded. The comparison principle (Lemma 4.7 in Hansevi [16]) implies that $H_V \bar{f}$ does not depend on the particular choice of extension \bar{f} . We can therefore define the p -harmonic extension for Lipschitz functions on the boundary by $H_V f := H_V \bar{f}$ if V is bounded.

Proposition 5.3. *If $\{f_j\}_{j=1}^\infty$ is a sequence of functions in $D^p(\Omega)$ that is decreasing q.e. in Ω to $f \in D^p(\Omega)$ and $\|g_{f_j-f}\|_{L^p(\Omega)} \rightarrow 0$ as $j \rightarrow \infty$, then Hf_j decreases to Hf locally uniformly in Ω .*

Proof. By the comparison principle (Lemma 4.7 in Hansevi [16]), it follows that $Hf_j \geq Hf_{j+1} \geq Hf$ in Ω for all $j = 1, 2, \dots$. Since Hf_j and Hf are the continuous solutions of the $\mathcal{K}_{f_j, Hf}$ - and $\mathcal{K}_{f, Hf}$ -obstacle problems, respectively, it follows from Theorem 3.2 that Hf_j decreases to Hf q.e. in Ω as $j \rightarrow \infty$.

Because Hf is continuous, and therefore locally bounded, Proposition 5.1 in Shanmugalingam [34] implies that $Hf_j \rightarrow Hf$ locally uniformly in Ω as $j \rightarrow \infty$. \square

In order to define Perron solutions, we need superharmonic functions. We follow Kinnunen–Martio [25], however, we use a slightly different, nevertheless equivalent, definition (see, e.g., Proposition 9.26 in Björn–Björn [3]).

Definition 5.4. We say that a function $u: \Omega \rightarrow (-\infty, \infty]$ is *superharmonic* in Ω if

- (a) u is lower semicontinuous;
- (b) u is not identically ∞ in any component of Ω ;
- (c) for every nonempty open set $V' \Subset \Omega$ and all $v \in \text{Lip}(\partial V')$, we have $H_{V'} v \leq u$ in V' whenever $v \leq u$ on $\partial V'$.

A function $u: \Omega \rightarrow [-\infty, \infty)$ is *subharmonic* in Ω if the function $-u$ is superharmonic.

6. Perron solutions

The standing assumptions are described at the end of Section 3. We make the convention from now on that the point at infinity, ∞ , belongs to the boundary $\partial\Omega$ if Ω is unbounded. Topological notions should therefore be understood with respect to the one-point compactification $X^* := X \cup \{\infty\}$.

Definition 6.1. Given a function $f: \partial\Omega \rightarrow \bar{\mathbb{R}}$, we let $\mathcal{U}_f(\Omega)$ be the set of all superharmonic functions u in Ω that are bounded below and such that

$$\liminf_{\Omega \ni y \rightarrow x} u(y) \geq f(x) \quad \text{for all } x \in \partial\Omega.$$

Then the *upper Perron solution* of f is defined by

$$\bar{P}_\Omega f(x) = \inf_{u \in \mathcal{U}_f(\Omega)} u(x), \quad x \in \Omega.$$

Similarly, we let $\mathcal{L}_f(\Omega)$ be the set of all subharmonic functions v in Ω that are bounded above and such that

$$\limsup_{\Omega \ni y \rightarrow x} v(y) \leq f(x) \quad \text{for all } x \in \partial\Omega,$$

and define the *lower Perron solution* of f by

$$\underline{P}_\Omega f(x) = \sup_{v \in \mathcal{L}_f(\Omega)} v(x), \quad x \in \Omega.$$

If $\bar{P}_\Omega f = \underline{P}_\Omega f$, then we let $P_\Omega f := \bar{P}_\Omega f$. Moreover, if $P_\Omega f$ is real-valued, then f is said to be *resolutive* (with respect to Ω). We often write Pf instead of $P_\Omega f$.

Immediate consequences of the above definition are that $\underline{P}f = -\bar{P}(-f)$ and that $\bar{P}f \leq \bar{P}h$ if $f \leq h$. It also follows that $\bar{P}f = \lim_{k \rightarrow \infty} \bar{P} \max\{f, -k\}$.

In each component of Ω , $\bar{P}f$ is either p -harmonic or identically $\pm\infty$, see, e.g., Björn–Björn [3] (their proof applies also to unbounded Ω). Thus Perron solutions are reasonable candidates for solutions of the Dirichlet problem.

The following theorem extends the comparison principle, which is fundamental for the nonlinear potential theory of superharmonic functions, and also plays an important role for the Perron method.

Theorem 6.2. *If u is superharmonic and v is subharmonic in Ω , then $v \leq u$ in Ω whenever*

$$\infty \neq \limsup_{\Omega \ni y \rightarrow x} v(y) \leq \liminf_{\Omega \ni y \rightarrow x} u(y) \neq -\infty \quad (6.1)$$

for all $x \in \partial\Omega$ (i.e., also for $x = \infty$ if Ω is unbounded).

Corollary 6.3. *If $f: \partial\Omega \rightarrow \bar{\mathbb{R}}$, then $\underline{P}f \leq \bar{P}f$.*

Proof of Theorem 6.2. Fix $\varepsilon > 0$. For each $x \in \partial\Omega$, it follows from (6.1) that

$$\liminf_{\Omega \ni y \rightarrow x} (u(y) - v(y)) \geq \liminf_{\Omega \ni y \rightarrow x} u(y) - \limsup_{\Omega \ni y \rightarrow x} v(y) \geq 0,$$

and hence there is an open set $U_x \subset X^*$ such that $x \in U_x$ and

$$u - v \geq -\varepsilon \quad \text{in } U_x \cap \Omega.$$

Let $\Omega_1, \Omega_2, \dots$ be open sets such that $\Omega_1 \Subset \Omega_2 \Subset \dots \Subset \Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Then

$$\bar{\Omega} \subset \bigcup_{k=1}^{\infty} \Omega_k \cup \bigcup_{x \in \partial\Omega} U_x.$$

Since $\bar{\Omega}$ is compact (with respect to the topology of X^*), there exist integers $k > 1/\varepsilon$ and N such that

$$\bar{\Omega} \subset \Omega_k \cup U_{x_1} \cup \dots \cup U_{x_N}.$$

It follows that $v \leq u + \varepsilon$ on $\partial\Omega_k$. Since v is upper semicontinuous (and does not take the value ∞), it follows that there is a decreasing sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \text{Lip}(\bar{\Omega}_k)$ such that $\varphi_j \rightarrow v$ on $\bar{\Omega}_k$ as $j \rightarrow \infty$ (see, e.g., Proposition 1.12 in Björn–Björn [3]).

Since $u + \varepsilon$ is lower semicontinuous, the compactness of $\partial\Omega_k$ shows that there exists an integer M such that $\varphi_M \leq u + \varepsilon$ on $\partial\Omega_k$, and, by (c) in Definition 5.4, also that $H_{\Omega_k} \varphi_M \leq u + \varepsilon$ in Ω_k . Similarly, $v \leq H_{\Omega_k} \varphi_M$, and thus $v \leq u + \varepsilon$ in Ω_k . Letting $\varepsilon \rightarrow 0$ (and hence letting $k \rightarrow \infty$) implies that $v \leq u$ in Ω . \square

7. Resolutivity of functions on $\partial\Omega$

In addition to the standing assumptions described at the end of Section 3, we assume that Ω is p -parabolic if Ω is unbounded (see Definition 4.1). For the convention about the point at infinity, see the beginning of Section 6.

When Björn–Björn–Shanmugalingam [9] extended the Perron method to the Mazurkiewicz boundary of bounded domains that are finitely connected at the boundary, they introduced a new capacity, $\overline{C}_p(\cdot; \Omega)$, adapted to the topology that connects the domain to its Mazurkiewicz boundary. They also used the new capacity to define $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous functions. By using $\overline{C}_p(\cdot; \Omega)$, which is smaller than the usual Sobolev capacity (see the appendix of [9]), we allow for perturbations on larger sets and we obtain resolvitivity for more functions.

Definition 7.1. The $\overline{C}_p(\cdot; \Omega)$ -capacity of a set $E \subset \overline{\Omega}$ is the number

$$\overline{C}_p(E; \Omega) := \inf_{u \in \mathcal{V}_E} \|u\|_{N^{1,p}(\Omega)}^p$$

where \mathcal{V}_E is the family of all functions $u \in N^{1,p}(\Omega)$ that satisfy both $u(x) \geq 1$ for all $x \in E \cap \Omega$ and

$$\liminf_{\Omega \ni y \rightarrow x} u(y) \geq 1 \quad \text{for all } x \in E \cap \partial\Omega. \quad (7.1)$$

When a property holds for all points except for points in a set of $\overline{C}_p(\cdot; \Omega)$ -capacity zero, it is said to hold $\overline{C}_p(\cdot; \Omega)$ -quasieverywhere (or $\overline{C}_p(\cdot; \Omega)$ -q.e. for short).

If $E \subset \Omega$, then condition (7.1) becomes empty and $\overline{C}_p(E; \Omega) = C_p(E; \Omega)$.

The capacity $\overline{C}_p(\cdot; \Omega)$ shares several properties with the Sobolev capacity, e.g., monotonicity and countable subadditivity. Moreover, $\overline{C}_p(\cdot; \Omega)$ is an outer capacity, i.e., if $E \subset \overline{\Omega}$, then

$$\overline{C}_p(E; \Omega) = \inf_{\substack{G \supset E \\ G \text{ relatively open in } \overline{\Omega}}} \overline{C}_p(G; \Omega).$$

These results are proved in Björn–Björn–Shanmugalingam [9] (a slightly modified version of their proof that $\overline{C}_p(\cdot; \Omega)$ is outer is valid in our setting as well).

To prove Theorem 7.5, we need the following version of Lemma 5.3 in Björn–Björn–Shanmugalingam [7].

Lemma 7.2. Assume that $\{U_k\}_{k=1}^\infty$ is a decreasing sequence of relatively open subsets of $\overline{\Omega}$ with $\overline{C}_p(U_k; \Omega) < 2^{-kp}$. Then there exists a sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ that decreases to zero q.e. in Ω , such that $\|\psi_j\|_{N^{1,p}(\Omega)} < 2^{-j}$ and $\psi_j \geq k-j$ in $U_k \cap \Omega$.

Proof. For each $k = 1, 2, \dots$, there exists a nonnegative function u_k such that $u_k = 1$ in $U_k \cap \Omega$ and $\|u_k\|_{N^{1,p}(\Omega)} < 2^{-k}$ because $\overline{C}_p(U_k; \Omega) < 2^{-kp}$. Letting

$$\psi_j = \sum_{k=j+1}^\infty u_k, \quad j = 1, 2, \dots,$$

yields a decreasing sequence of nonnegative functions such that $\|\psi_j\|_{N^{1,p}(\Omega)} < 2^{-j}$ and $\psi_j \geq k-j$ in $U_k \cap \Omega$. Corollary 3.9 in Shanmugalingam [32] implies the existence of a subsequence of $\{\psi_j\}_{j=1}^\infty$ that converges to zero q.e. in Ω , and since $\{\psi_j\}_{j=1}^\infty$ is nonnegative and decreasing, this shows that $\{\psi_j\}_{j=1}^\infty$ decreases to zero q.e. in Ω . \square

Definition 7.3. Let f be an extended real-valued function defined on $\overline{\Omega} \setminus \{\infty\}$. We say that f is $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous on $\overline{\Omega} \setminus \{\infty\}$ if for every $\varepsilon > 0$ there is a relatively open subset U of $\overline{\Omega} \setminus \{\infty\}$ with $\overline{C}_p(U; \Omega) < \varepsilon$ such that the restriction of f to $(\overline{\Omega} \setminus \{\infty\}) \setminus U$ is continuous and real-valued.

Since the $\overline{C}_p(\cdot; \Omega)$ -capacity is smaller than the Sobolev capacity (which is used to define quasicontinuity), it follows that quasicontinuous functions are also $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous.

Proposition 7.4. *If $f: \overline{\Omega} \setminus \{\infty\} \rightarrow \overline{\mathbb{R}}$ is a function such that $f = 0$ q.e. on $\partial\Omega \setminus \{\infty\}$ and $f|_{\Omega} \in D_0^p(\Omega)$, then f is $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous on $\overline{\Omega} \setminus \{\infty\}$.*

Proof. Extend f to X by letting f be equal to zero outside $\overline{\Omega}$ so that $f \in D^p(X)$. Then $f \in N_{\text{loc}}^{1,p}(X)$ by Proposition 4.14 in Björn–Björn [3], and hence Theorem 1.1 in Björn–Björn–Shanmugalingam [8] asserts that f is quasicontinuous on X , and therefore $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous on $\overline{\Omega} \setminus \{\infty\}$. \square

The following is the main result of this paper.

Theorem 7.5. *Assume that $f: \overline{\Omega} \rightarrow \overline{\mathbb{R}}$ is $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous on $\overline{\Omega} \setminus \{\infty\}$ and such that $f|_{\Omega} \in D^p(\Omega)$, which in particular hold if $f \in D^p(X)$. Then f is resolutive with respect to Ω and $Pf = Hf$.*

To see that p -parabolicity is needed in Theorem 7.5 if Ω is unbounded, let $n > p$ and let $\Omega = \mathbb{R}^n \setminus \overline{B}$, where B is the open unit ball centered at the origin. Then Ω is p -hyperbolic. Furthermore, let

$$f(x) = |x|^{\frac{p-n}{p-1}}, \quad x \in \overline{\Omega}.$$

Then f satisfies the hypothesis of Theorem 7.5. Because $f \equiv 1$ on ∂B and the p -harmonic extension does not consider the point at infinity, it is clear that $Hf \equiv 1$. However, $Pf \equiv f$, since f is in fact p -harmonic (it is easy to verify that f is a solution of the p -Laplace equation (1.1)) and continuous on $\overline{\Omega}$, and hence $f \in \mathcal{U}_f(\Omega)$ and $f \in \mathcal{L}_f(\Omega)$, which implies that $f \leq \underline{P}f \leq \overline{P}f \leq f$.

Proof of Theorem 7.5. Suppose that Ω is unbounded and p -parabolic. Let $\{K_j\}_{j=1}^{\infty}$ be an increasing sequence of compact sets such that $K_1 \subseteq K_2 \subseteq \dots \subseteq \Omega = \bigcup_{j=1}^{\infty} K_j$ and let $x_0 \in X$. For each $j = 1, 2, \dots$, we can find a function $u_j \in D^p(\Omega)$ such that $\chi_{K_j} \leq u_j \leq 1$, $u_j = 0$ in $\Omega \setminus B_j$ for some ball $B_j \supset K_j$ centered at x_0 , and

$$\|g_{u_j}\|_{L^p(\Omega)} < 2^{-j}. \quad (7.2)$$

Let

$$\xi_j = \sum_{k=j}^{\infty} (1 - u_k), \quad j = 1, 2, \dots \quad (7.3)$$

Then $\xi_j \geq 0$ and

$$\|g_{\xi_j}\|_{L^p(\Omega)} \leq \sum_{k=j}^{\infty} \|g_{u_k}\|_{L^p(\Omega)} < \sum_{k=j}^{\infty} 2^{-k} = 2^{1-j}. \quad (7.4)$$

Let $\Omega_j = \bigcup_{n=1}^j B_n \cap \Omega$, $j = 1, 2, \dots$. Then $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Since $u_j = 0$ in $\Omega \setminus \Omega_j$, it is easy to see that

$$\lim_{\Omega \ni y \rightarrow \infty} \xi_j(y) = \infty \quad \text{for all } j = 1, 2, \dots \quad (7.5)$$

Furthermore, since $\{\xi_j\}_{j=1}^{\infty}$ is decreasing and $\xi_j = 0$ on K_j for each $j = 1, 2, \dots$, it follows that $\{\xi_j\}_{j=1}^{\infty}$ decreases to zero in Ω .

On the other hand, if Ω is bounded, then we let $\xi_j \equiv 0$ in Ω , $j = 1, 2, \dots$.

The p -harmonic extension Hf is $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous on $\overline{\Omega} \setminus \{\infty\}$ (when we consider Hf to be equal to f on $\partial\Omega$), since Proposition 7.4 asserts that $Hf - f$ is $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous on $\overline{\Omega} \setminus \{\infty\}$ as $(Hf - f)|_{\Omega} \in D_0^p(\Omega)$. We can therefore find a decreasing sequence $\{U_k\}_{k=1}^{\infty}$ of relatively open subsets of $\overline{\Omega} \setminus \{\infty\}$ with $\overline{C}_p(U_k; \Omega) < 2^{-kp}$ and such that the restriction of Hf to $(\overline{\Omega} \setminus \{\infty\}) \setminus U_k$ is continuous.

Now we derive that $\bar{P}f \leq Hf$ q.e. in Ω if f is bounded from below. Without loss of generality, we may as well assume that $f \geq 0$. Then the comparison principle (Lemma 4.7 in Hansevi [16]) implies that $Hf \geq 0$ in Ω .

Consider the sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ given by Lemma 7.2, and define $h_j: \Omega \rightarrow [0, \infty]$ by letting

$$h_j = Hf + \xi_j + \psi_j, \quad j = 1, 2, \dots$$

Then $h_j \in D^p(\Omega)$ and $\{h_j\}_{j=1}^\infty$ decreases to Hf q.e. in Ω .

Let φ_j be the lsc-regularized solution of the \mathcal{K}_{h_j, h_j} -obstacle problem, $j = 1, 2, \dots$. By (7.4) and Lemma 7.2,

$$\|g_{h_j - Hf}\|_{L^p(\Omega)} \leq \|g_{\xi_j}\|_{L^p(\Omega)} + \|g_{\psi_j}\|_{L^p(\Omega)} < 2^{1-j} + 2^{-j} \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

and as Hf is a solution of the $\mathcal{K}_{Hf, Hf}$ -obstacle problem, it follows from Theorem 3.2 that $\{\varphi_j\}_{j=1}^\infty$ decreases to Hf q.e. in Ω . This will be used later in the proof.

Next we show that

$$\liminf_{\Omega \ni y \rightarrow x} \varphi_j(y) \geq f(x) \quad \text{for all } x \in \partial\Omega. \quad (7.6)$$

Fix a positive integer m and let $\varepsilon = 1/m$. By Lemma 7.2,

$$h_j(y) \geq \psi_j(y) \geq m \quad \text{for all } y \in U_{m+j} \cap \Omega. \quad (7.7)$$

Let $x \in \partial\Omega \setminus \{\infty\}$. If $x \notin U_{m+j}$, then as the restriction of Hf to $(\bar{\Omega} \setminus \{\infty\}) \setminus U_{m+j}$ is continuous, there is a relative neighborhood $V_x \subset \bar{\Omega} \setminus \{\infty\}$ of x such that

$$h_j(y) \geq Hf(y) \geq Hf(x) - \varepsilon = f(x) - \varepsilon \quad \text{for all } y \in (V_x \cap \Omega) \setminus U_{m+j}. \quad (7.8)$$

By combining (7.7) and (7.8), we see that for $x \in (\partial\Omega \setminus \{\infty\}) \setminus U_{m+j}$,

$$h_j(y) \geq \min\{f(x) - \varepsilon, m\} \quad \text{for all } y \in V_x \cap \Omega. \quad (7.9)$$

On the other hand, if $x \in U_{m+j}$, then we let $V_x = U_{m+j}$, and see that (7.9) holds also in this case due to (7.7). Because $\varphi_j \geq h_j$ q.e. in Ω and φ_j is lsc-regularized, it follows that

$$\varphi_j(y) \geq \min\{f(x) - \varepsilon, m\} \quad \text{for all } y \in V_x \cap \Omega,$$

and hence

$$\liminf_{\Omega \ni y \rightarrow x} \varphi_j(y) \geq \min\{f(x) - \varepsilon, m\}.$$

Letting $m \rightarrow \infty$ (and thus letting $\varepsilon \rightarrow 0$) establishes that

$$\liminf_{\Omega \ni y \rightarrow x} \varphi_j(y) \geq f(x) \quad \text{for all } x \in \partial\Omega \setminus \{\infty\}.$$

Finally, if Ω is unbounded, then $\varphi_j \geq h_j$ q.e. in Ω and $h_j \geq \xi_j$ everywhere in Ω . From the lsc-regularity of φ_j and (7.5), it follows that

$$\liminf_{\Omega \ni y \rightarrow \infty} \varphi_j(y) \geq \lim_{\Omega \ni y \rightarrow \infty} \xi_j(y) = \infty,$$

and hence we have shown that (7.6) holds.

Since φ_j is an lsc-regularized superminimizer, Proposition 7.4 in Kinnunen–Martio [25] asserts that φ_j is superharmonic. As φ_j is bounded from below and (7.6) holds, it follows that $\varphi_j \in \mathcal{U}_f(\Omega)$, and hence we know that $\bar{P}f \leq \varphi_j$, $j = 1, 2, \dots$. Because $h_j \in D^p(\Omega)$ and $\{h_j\}_{j=1}^\infty$ decreases to Hf q.e. in Ω , $\|g_{h_j - Hf}\|_{L^p(\Omega)} \rightarrow 0$ as $j \rightarrow \infty$, and Hf is a solution of the $\mathcal{K}_{Hf, Hf}$ -obstacle problem, it follows from

Theorem 3.2 that $\{\varphi_j\}_{j=1}^\infty$ decreases to Hf q.e. in Ω . We therefore conclude that $\bar{P}f \leq Hf$ q.e. in Ω (provided that f is bounded from below).

Now we remove the extra assumption of f being bounded from below, and let $f_k = \max\{f, -k\}$, $k = 1, 2, \dots$. Then $\{f_k\}_{k=1}^\infty$ is decreasing to f . Proposition 4.14 in Björn–Björn [3] implies that $f \in L_{\text{loc}}^p(\Omega)$. Hence $\mu(\{x \in \Omega : |f(x)| = \infty\}) = 0$, and therefore $\chi_{\{x \in \Omega : f(x) < -k\}} \rightarrow 0$ a.e. in Ω as $k \rightarrow \infty$. Since

$$g_{f_k-f} = g_{\max\{0, -f-k\}} = gf\chi_{\{x \in \Omega : f(x) < -k\}} \quad \text{a.e. in } \Omega,$$

implies that $g_{f_k-f} \rightarrow 0$ a.e. in Ω as $k \rightarrow \infty$, and because $g_f \in L^p(\Omega)$ and

$$g_{f_k-f} \leq g_{f_k} + g_f \leq 2g_f \quad \text{a.e. in } \Omega,$$

it follows by dominated convergence that $g_{f_k-f} \rightarrow 0$ in $L^p(\Omega)$ as $k \rightarrow \infty$. Thus Proposition 5.3 asserts that

$$Hf_k \rightarrow Hf \quad \text{in } \Omega \text{ as } k \rightarrow \infty.$$

Since f_k is bounded from below, it follows that

$$\bar{P}f = \lim_{k \rightarrow \infty} \bar{P}f_k \leq \lim_{k \rightarrow \infty} Hf_k = Hf \quad \text{q.e. in } \Omega.$$

As both $\bar{P}f$ and Hf are continuous, we conclude that $\bar{P}f \leq Hf$ everywhere in Ω . By Corollary 6.3, it follows that

$$\bar{P}f \leq Hf = -H(-f) \leq -\bar{P}(-f) = Pf \leq \bar{P}f \quad \text{in } \Omega,$$

which implies that f is resolutive and that $Pf = Hf$. \square

Perron solutions are invariant under perturbation of the function on a set of capacity zero.

Theorem 7.6. *Assume that $f: \bar{\Omega} \rightarrow \bar{\mathbb{R}}$ is $\bar{C}_p(\cdot; \Omega)$ -quasicontinuous on $\bar{\Omega} \setminus \{\infty\}$ and such that $f|_\Omega \in D^p(\Omega)$, which in particular hold if $f \in D^p(X)$. Assume also that $h: \partial\Omega \rightarrow \bar{\mathbb{R}}$ is zero $\bar{C}_p(\cdot; \Omega)$ -q.e. on $\partial\Omega \setminus \{\infty\}$. Then $f + h$ is resolutive with respect to Ω and $P(f + h) = Pf$.*

Proof. Extend h by zero in Ω and let $E = \{x \in \bar{\Omega} : h(x) \neq 0\}$. Since $\bar{C}_p(\cdot; \Omega)$ is an outer capacity, it follows that given $\varepsilon > 0$, we can find a relatively open subset U of $\bar{\Omega} \setminus \{\infty\}$ with $\bar{C}_p(U; \Omega) < \varepsilon$ and such that $E \subset U$, and hence h is $\bar{C}_p(\cdot; \Omega)$ -quasicontinuous on $\bar{\Omega} \setminus \{\infty\}$. The subadditivity of the $\bar{C}_p(\cdot; \Omega)$ -capacity implies that this is true also for $f + h$.

Since $f + h = f$ in Ω and $f|_\Omega \in D^p(\Omega)$, we know that $H(f + h) = Hf$. We complete the proof by applying Theorem 7.5 to both f and $f + h$, which shows that $f + h$ is resolutive and that

$$P(f + h) = H(f + h) = Hf = Pf. \quad \square$$

The following uniqueness result is a direct consequence of Theorem 7.6.

Corollary 7.7. *Assume that u is bounded and p -harmonic in Ω . Assume also that $f: \bar{\Omega} \rightarrow \bar{\mathbb{R}}$ is $\bar{C}_p(\cdot; \Omega)$ -quasicontinuous on $\bar{\Omega} \setminus \{\infty\}$ and such that $f|_\Omega \in D^p(\Omega)$. Then $u = Pf$ in Ω whenever there exists a set $E \subset \partial\Omega$ with $\bar{C}_p(E \setminus \{\infty\}; \Omega) = 0$ such that*

$$\lim_{\Omega \ni y \rightarrow x} u(y) = f(x) \quad \text{for all } x \in \partial\Omega \setminus E.$$

Proof. Since $\overline{C}_p(E \setminus \{\infty\}; \Omega) = 0$, Theorem 7.6 applies to f and $h := \infty \chi_E$ (and clearly also to f and $-h$), and because $u \in \mathcal{U}_{f-h}(\Omega)$ and $u \in \mathcal{L}_{f+h}(\Omega)$ (since u is bounded), it follows that

$$u \leq \underline{P}(f+h) = P(f+h) = Pf = P(f-h) = \overline{P}(f-h) \leq u \quad \text{in } \Omega. \quad \square$$

The obtained resolvitivity results can now be extended to continuous functions. Björn–Björn–Shanmugalingam [7],[9] proved the following result for bounded domains.

Theorem 7.8. *If $f \in C(\partial\Omega)$ and $h: \partial\Omega \rightarrow \overline{\mathbb{R}}$ is zero $\overline{C}_p(\cdot; \Omega)$ -q.e. on $\partial\Omega \setminus \{\infty\}$, then f and $f+h$ are resolutive with respect to Ω and $P(f+h) = Pf$.*

Proof. We start by choosing a point $x_0 \in \partial\Omega$. If Ω is unbounded, then we let $x_0 = \infty$. Let $\alpha = f(x_0) \in \mathbb{R}$ and let j be a positive integer. Since $f \in C(\partial\Omega)$, there exists a compact set $K_j \subset X$ such that $|f(x) - \alpha| < 1/3j$ for all $x \in \partial\Omega \setminus K_j$. Let

$$K'_j = \{x \in X : \text{dist}(x, K_j) \leq 1\}.$$

We can find functions $\varphi_j \in \text{Lip}_c(X)$ such that $|\varphi_j - f| \leq 1/3j$ on $\partial\Omega \cap K'_j$. Let $f_j = (\varphi_j - \alpha)\eta_j + \alpha$, where

$$\eta_j(x) := \begin{cases} 1, & x \in K_j, \\ 1 - \text{dist}(x, K_j), & x \in K'_j \setminus K_j, \\ 0, & x \in X \setminus K'_j. \end{cases}$$

Since f_j is Lipschitz on X and $f_j = \alpha$ outside K'_j , it follows that $f_j \in D^p(X)$.

Let $x \in \partial\Omega$. Then $|f_j(x) - f(x)| \leq 1/3j$ whenever $x \notin K'_j \setminus K_j$. Otherwise it follows that

$$\begin{aligned} |f_j(x) - f(x)| &= |(\varphi_j(x) - \alpha)\eta_j(x) + \alpha - f(x)| \leq |\varphi_j(x) - \alpha| + |\alpha - f(x)| \\ &\leq |\varphi_j(x) - f(x)| + 2|f(x) - \alpha| < \frac{1}{j}, \end{aligned}$$

and thus we know that $f - 1/j \leq f_j \leq f + 1/j$ on $\partial\Omega$. It follows directly from Definition 6.1 that $\underline{P}f - 1/j \leq \underline{P}f_j \leq \underline{P}f + 1/j$, and we also get corresponding inequalities for $\overline{P}f_j$, $\underline{P}(f_j + h)$, and $\overline{P}(f_j + h)$.

Theorem 7.6 asserts that f_j and $f_j + h$ are resolutive and that $P(f_j + h) = Pf_j$. It follows that

$$\overline{P}f - \frac{1}{j} \leq \overline{P}f_j = \underline{P}f_j \leq \underline{P}f + \frac{1}{j}. \quad (7.10)$$

Applying Corollary 6.3 to (7.10) yields $0 \leq \overline{P}f - \underline{P}f \leq 2/j$. Letting $j \rightarrow \infty$ shows that f is resolutive. Similarly, we can see that also $f+h$ is resolutive.

Finally, we have

$$P(f+h) - Pf = \overline{P}(f+h) - \underline{P}f \leq \overline{P}(f_j+h) + \frac{1}{j} - \left(\underline{P}f_j - \frac{1}{j}\right) = \frac{2}{j}. \quad (7.11)$$

Interchanging $\overline{P}(f+h)$ and $\underline{P}f$ with $\underline{P}(f+h)$ and $\overline{P}f$, respectively, in (7.11) yields $P(f+h) - Pf \geq -2/j$, and hence $|P(f+h) - Pf| < 2/j$. Letting $j \rightarrow \infty$ shows that $P(f+h) = Pf$. \square

We conclude this paper with the following uniqueness result, corresponding to Corollary 7.7, that follows directly from Theorem 7.8. The proof is identical to the proof of Corollary 7.7, except for applying Theorem 7.8 (instead of Theorem 7.6).

Corollary 7.9. *Assume that u is bounded and p -harmonic in Ω . If $f \in C(\partial\Omega)$ and there is a set $E \subset \partial\Omega$ with $\overline{C}_p(E \setminus \{\infty\}; \Omega) = 0$ such that*

$$\lim_{\Omega \ni y \rightarrow x} u(y) = f(x) \quad \text{for all } x \in \partial\Omega \setminus E,$$

then $u = Pf$ in Ω .

References

1. BJÖRN, A., A weak Kellogg property for quasiminimizers, *Comment. Math. Helv.* **81** (2006), 809–825.
2. BJÖRN, A., The Dirichlet problem for p -harmonic functions on the topologist's comb, *Math. Z.* **279** (2015), 389–405.
3. BJÖRN, A. and BJÖRN, J., *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics **17**, European Math. Soc., Zürich, 2011.
4. BJÖRN, A. and BJÖRN, J., Obstacle and Dirichlet problems on arbitrary non-open sets, and fine topology, *Rev. Mat. Iberoam.* **31** (2015), 161–214.
5. BJÖRN, A., BJÖRN, J. and PARVIAINEN, M., Lebesgue points and the fundamental convergence theorem for superharmonic functions on metric spaces, *Rev. Mat. Iberoam.* **26** (2010), 147–174.
6. BJÖRN, A., BJÖRN, J. and SHANMUGALINGAM, N., The Dirichlet problem for p -harmonic functions on metric spaces, *J. Reine Angew. Math.* **556** (2003), 173–203.
7. BJÖRN, A., BJÖRN, J. and SHANMUGALINGAM, N., The Perron method for p -harmonic functions in metric spaces, *J. Differential Equations* **195** (2003), 398–429.
8. BJÖRN, A., BJÖRN, J. and SHANMUGALINGAM, N., Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces, *Houston J. Math.* **34** (2008), 1197–1211.
9. BJÖRN, A., BJÖRN, J. and SHANMUGALINGAM, N., The Dirichlet problem for p -harmonic functions with respect to the Mazurkiewicz boundary, *J. Differential Equations* **259** (2015), 3078–3114.
10. BJÖRN, A., BJÖRN, J. and SJÖDIN, T., The Dirichlet problem for p -harmonic functions with respect to arbitrary compactifications (2016, preprint), [arXiv:1604.08731](https://arxiv.org/abs/1604.08731).
11. BRELOT, M., Familles de Perron et problème de Dirichlet, *Acta Litt. Sci. Szeged* **9** (1939), 133–153.
12. ESTEP, D. and SHANMUGALINGAM, N., Geometry of prime end boundary and the Dirichlet problem for bounded domains in metric measure spaces, *Potential Anal.* **42** (2015), 335–363.
13. FARNANA, Z., Continuous dependence on obstacles for the double obstacle problem on metric spaces, *Nonlinear Anal.* **73** (2010), 2819–2830.
14. GRANLUND, S., LINDQVIST, P. and MARTIO, O., Note on the PWB-method in the non-linear case, *Pacific J. Math.* **125** (1986), 381–395.
15. HAJLASZ, P. and KOSKELA, P., *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000).
16. HANSEVI, D., The obstacle and Dirichlet problems associated with p -harmonic functions in unbounded sets in \mathbb{R}^n and metric spaces, *Ann. Acad. Sci. Fenn. Math.* **40** (2015), 89–108.
17. HEINONEN, J., *Lectures on Analysis on Metric Spaces*, Universitext, Springer-Verlag, New York, 2001.
18. HEINONEN, J., KILPELÄINEN, T. and MARTIO, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, 2nd ed., Dover, Mineola, NY, 2006.
19. HEINONEN, J. and KOSKELA, P., From local to global in quasiconformal structures, *Proc. Natl. Acad. Sci. USA* **93** (1996), 554–556.
20. HEINONEN, J. and KOSKELA, P., Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.
21. HEINONEN, J., KOSKELA, P., SHANMUGALINGAM, N. and TYSON, J. T., *Sobolev Spaces on Metric Measure Spaces*, New Mathematical Monographs **27**, Cambridge University Press, Cambridge, 2015.

22. HOLOPAINEN, I. and SHANMUGALINGAM, N., Singular functions on metric measure spaces, *Collect. Math.* **53** (2002), 313–332.
23. KILPELÄINEN, T., Potential theory for supersolutions of degenerate elliptic equations, *Indiana Univ. Math. J.* **38** (1989), 253–275.
24. KINNUNEN, J. and MARTIO, O., The Sobolev capacity on metric spaces, *Ann. Acad. Sci. Fenn. Math.* **21** (1996), 367–382.
25. KINNUNEN, J. and MARTIO, O., Nonlinear potential theory on metric spaces, *Illinois J. Math.* **46** (2002), 857–883.
26. KINNUNEN, J. and SHANMUGALINGAM, N., Polar sets on metric spaces, *Trans. Amer. Math. Soc.* **358** (2006), 11–37.
27. KOSKELA, P. and MACMANUS, P., Quasiconformal mappings and Sobolev spaces, *Studia Math.* **131** (1998), 1–17.
28. MALÝ, J. and ZIEMER, W. P., *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs **51**, American Math. Soc., Providence, RI, 1997.
29. PERRON, O., Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$, *Math. Z.* **18** (1923), 42–54.
30. REMAK, R., Über potentialkonvexe Funktionen, *Math. Z.* **20** (1924), 126–130.
31. RUDIN, W., *Functional Analysis*, McGraw-Hill, New York, 1991.
32. SHANMUGALINGAM, N., Newtonian spaces: an extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoam.* **16** (2000), 243–279.
33. SHANMUGALINGAM, N., Harmonic functions on metric spaces, *Illinois J. Math.* **45** (2001), 1021–1050.
34. SHANMUGALINGAM, N., Some convergence results for p -harmonic functions on metric measure spaces, *Proc. London Math. Soc.* **87** (2003), 226–246.